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LECTURE 1

Some Paradoxes in Elementary Electromagnetism

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Elementary electricity and magnetism generally seem to be pretty clear-cut subjects with nice tidy laws all fitting neatly together. We have simple basic laws such as Coulomb's law for the force between electric charges, Ampère's law for the force between electric currents, the Biot-Savart law for the magnetic field of an electric current. On a slightly higher level we can add Gauss's law for the magnetic field, Ampère's circuital law for the line integral of the magnetic field and Faraday's law of electromagnetic induction. It is one of the most coherent topics in the whole of physics. It is almost paradoxical to suggest that this beautiful subject could contain any paradoxes.

Let us start by being sure we all mean the same thing by the word paradox. My Concise Oxford Dictionary gives three different meanings, two of which are appropriate to tonight's lecture: "seemingly absurd though perhaps really well-founded statement" and "thing conflicting with preconceived notions of what is reasonable or possible". The phenomena which I shall present tonight all have in common the fact that they regularly cause people to think there is something ambiguous, if not plain wrong, in electromagnetic theory, and it is my job to resolve these paradoxes and show that they do not bring the edifice of physics crumbling to the ground.

First let us consider two like electric charges moving at the same speed along parallel paths. They constitute a pair of parallel electric currents and according to Ampère's law they will attract each other but to an observer moving with the same velocity as the charges, they are a pair of stationary like charges and according to Coulomb's law they will repel each other. Who is right? I want to start by considering not two point charges but rather two long straight charged rods, with a charge per unit length of \( +\lambda \). The electric field of one such rod is readily calculated to be

\[
E = \frac{\lambda}{2\pi \varepsilon_0 r}
\]
directly away from the rod. This follows quite simply from the fact that since we can represent an electric field by lines of force the number of which per unit area is equal to the magnitude of the field. The field at the surface of a sphere of radius $r$ with charge $q$ at its centre is given by

$$E = \frac{q}{4\pi \varepsilon_0 r^2} = \text{No. of lines leaving } q \over \text{surface area of sphere} = \frac{\text{No. of lines leaving } q}{4\pi r^2}$$

$. \text{ No. of lines leaving } q \text{ is } \frac{q}{\varepsilon_0}$.

Since the total charge on unit length of our charged rod is $\lambda$, the number of lines leaving unit length is $\frac{\lambda}{\varepsilon_0}$ and by symmetry arguments they must be radial. Hence at radius $r$, $\frac{\lambda}{\varepsilon_0}$ lines pass through the curved surface of a cylinder of radius $r$ and unit length.

$$E = \text{No. of lines per unit area} = \frac{\lambda/\varepsilon_0}{2\pi r} = \frac{\lambda}{2\pi \varepsilon_0 r}.$$

The force of repulsion between the two rods is therefore

$$f_e = \frac{\lambda^2}{2\pi \varepsilon_0 r} \text{ per unit length}.$$

Regarded as a pair of currents the rods are currents of magnitude given by

$$i = \lambda v \text{ where } v \text{ is the speed of rods.}$$

The force of attraction between these two parallel currents is, by Ampère's law,

$$f = \frac{\mu_0 i^2}{2\pi r} = \frac{\mu_0 \lambda^2 v^2}{2\pi r} \text{ per unit length}.$$
The resultant force of repulsion is

\[ f = \frac{f_e}{r} - \frac{f_m}{r} \quad \text{per unit length} \]

\[ = \lambda^2 \frac{\mu_0 \lambda^2 v^2}{2\pi \varepsilon_0 r} - \frac{\mu_0 \varepsilon_0 v^2}{2\pi r} \]

\[ = \lambda^2 \frac{\lambda}{2\pi \varepsilon_0 r} \left[ 1 - \mu_0 \varepsilon_0 \frac{v^2}{c^2} \right] \]

\[ = \lambda^2 \frac{\lambda}{2\pi \varepsilon_0 r} \left( 1 - \frac{v^2}{c^2} \right) \text{ where } c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \text{speed of light} \]

Since \( v < c \), the force is always one of repulsion. Qualitatively, this is the result the observer travelling with the rod would get, but how about quantitatively? The observer we have just considered sees the rods moving with speed \( v \) in the direction of their length, and therefore Lorentz contracted by the factor \( \sqrt{1 - \frac{v^2}{c^2}} \). The charge per unit length is therefore different for the two observers, and related by

\[ \lambda' = \lambda \sqrt{1 - \frac{v^2}{c^2}} \]

where \( \lambda' \) is the value in the rest frame of the two rods. In this reference frame the electric repulsion is

\[ \frac{f_e'}{r} = \lambda^2 \frac{\mu_0 \lambda^2 v^2}{2\pi \varepsilon_0 r} \quad \text{per unit length} \]

\[ = \lambda^2 \frac{\lambda}{2\pi \varepsilon_0 r} \left( 1 - \frac{v^2}{c^2} \right) \]

and the magnetic force is zero. Hence \( f' = f \). Both observers get the same answer for the force per unit length. However that isn't the whole story,
because they do not agree as to what section of a rod is unit length. The observer who sees the rods as moving will say a section of rod of length $\ell'$ according to the rest frame observer is really of length $\ell$, where

$$\ell = \ell' \sqrt{1 - \frac{v^2}{c^2}} \quad \text{(the Lorentz contraction again)}$$

Hence if we consider, not forces per unit length but forces on individual charges, we find

$$F' = f' \ell'$$

and

$$F = f \ell = f' \ell' \sqrt{1 - \frac{v^2}{c^2}} = F' \sqrt{1 - \frac{v^2}{c^2}}$$

So now we have disagreement concerning forces, but force is rate of change of momentum and momentum concerns what actually happens to a body so let us talk about that instead. If a force $F$ acts on the body for some time it produces a change of momentum $\Delta p = F \Delta t$. For our two observers we have

$$F = F' \sqrt{1 - \frac{v^2}{c^2}}$$

but we also have

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{(the Einstein time dilation)}$$

Hence $F t = F' t'$ and $\Delta p = \Delta p'$. 
The two observers are in complete agreement as to what happens to the motion of the two charges.

Now if we return to the original question which concerned two single charges, we see that the observer who sees them at rest says they repel with a force \( F' = F'_e \), and the observer who sees them as electric currents says they repel with a force \( F = F_e - F_m \)

\[
= \frac{q^2}{4\pi\varepsilon_0 r^2} - \frac{\mu_0 q^2 v^2}{4\pi r^2} = \frac{q^2}{4\pi\varepsilon_0 r^2} (1 - \frac{\mu_0 e_0 v^2}{c^2})
\]

\[
= F_e \left(1 - \frac{v^2}{c^2}\right)
\]

But \( F = F_e \sqrt{1 - \frac{v^2}{c^2}} \)

so \( F'_e \sqrt{1 - \frac{v^2}{c^2}} = F_e \left(1 - \frac{v^2}{c^2}\right) \)

and

\[
F_e = \frac{F'_e}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{q^2}{4\pi\varepsilon_0 r^2} \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}}
\]

and the electric fields \( E = \frac{F}{q} \) in \( \Sigma \) and \( E' = \frac{F'}{q} \) in \( \Sigma' \)

are related by \( E = \sqrt{\frac{E'}{1 - \frac{v^2}{c^2}}} > E' \)

If, instead of considering two charges lying on a line perpendicular to the direction of motion of \( \Sigma' \) relative to \( \Sigma \), we place the charges on a line in the direction of this motion, it can be shown (though I am not going to show it here) that \( F = F' \). The two observers now agree on the magnitude of the force between the charges but they disagree on the separation!

\[
r = r' \sqrt{1 - \frac{v^2}{c^2}} \quad \text{(the Lorentz contraction again)}
\]
Coulomb's law undoubtedly holds in the rest frame Σ',

\[ F_{e}' = \frac{q^2}{4\pi \varepsilon_0 r'^2} \]

and \[ E' = \frac{q}{4\pi \varepsilon_0 r'^2} \]

In Σ, however, \[ F_e = F_e' = \frac{q^2(1 - \frac{v^2}{c^2})}{4\pi \varepsilon_0 r^2} \] and \[ E = \frac{q}{4\pi \varepsilon_0 r^2} (1 - \frac{v^2}{c^2}) < E' \]

We have therefore now the result that the electric field of a moving charge is increased in a plane perpendicular to the direction of its motion and decreased in the direction of its motion. The field lines of a moving charge would there look like this.

![Diagram](attachment:image.png)

direction of motion

We have come a long way from our original paradox question and have ended up in the realm of special relativity, but this has served to show that magnetic forces and electrostatic forces are not separate entities and this has been the physics underlying the paradox.

I want to present another paradox now which is even tougher than the one we have just talked about. It concerns the Biot-Savart Law which always seems very simple and straightforward. Let me start by writing it down.
\[ dB = \frac{\mu_0}{4\pi} \frac{i \, d\ell \, \sin \theta}{r^2} \]

dB, the magnetic field due to the current element \( i \, d\ell \) is perpendicular to the plane containing \( d\ell \) and \( r \).

This formula, which gives the magnetic field due to a current element, is used extensively in calculation of the magnetic fields due to currents in circuits, by adding (integrating) the contributions of all the current elements in the circuit; and it gives the right answer! Where then is the paradox?

Consider two current elements and, for the sake of being specific, we shall consider them to belong to different circuits (although this is irrelevant to what follows). Suppose they are in the same plane and oriented like this:

\[ \theta_1 = 0 \] \( i_1 \, d\ell_1 \)

\[ \theta_2 = 90^\circ \] \( i_2 \, d\ell_2 \)

The magnetic field at \( i_1 \, d\ell_1 \) due to \( i_2 \, d\ell_2 \) is, according to Biot & Savart

\[ dB_1 = \frac{\mu_0}{4\pi} \frac{i_2 \, d\ell_2 \sin \theta_2}{r^2} = \frac{\mu_0}{4\pi} \frac{i_2 \, d\ell_2}{r^2} \] out of the page

The resulting force on \( i_1 \, d\ell_1 \) is \( d^2 F_1 = i \, d\ell_1 \, dB \) down the page.

However the field at \( i_2 \, d\ell_2 \) due to \( i_1 \, d\ell_1 \) is

\[ dB_2 = \frac{\mu_0}{4\pi} \frac{i_1 \, d\ell_1 \sin \theta_1}{r^2} = 0 \]
and the resulting force on $i_2 d\ell_2$ is zero.
Hence $i_2 d\ell_2$ exerts a force on $i_1 d\ell_1$, but $i_1 d\ell_1$ does not exert a force on $i_2 d\ell_2$ in clear contravention of Newton's third law! To be told that we never deal with isolated current elements but only with closed circuits and that when the Biot-Savart law is integrated around the circuit 1, of which $i_1 d\ell_1$ is part, and around circuit 2, of which $i_2 d\ell_2$ is part, the resultant forces experienced by the complete circuits do indeed turn out to be equal and opposite does little to satisfy our emotional desire to see the laws of physics obeyed on the microscopic as well as the macroscopic scale.

Actually we can get rid of the "rest of the circuit" by considering a pair of moving charges instead of a pair of current elements i.e. $i_1 d\ell_1$ becomes $q_1 v_1$ and $i_2 d\ell_2$ becomes $q_2 v_2$. We now have the paradox right back again in its original form, with no macroscopic "out" at our disposal. How are we going to get out of this one? The short answer is that we're not — we shall identify the false reasoning which leads to it but we shall not go through all the very complex quantitative discussion of the situation.

Basically the paradox arises because from the whole story of charges, currents, and fields which describes this interaction we have focussed our attention on one part of the interaction and completely ignored the rest. The problem really concerns the total electromagnetic interaction between two moving charges and their electric and magnetic fields. Throughout the encounter we naturally require the total momentum of the system to remain constant.

The interaction of the electric and magnetic fields of the charges gives rise to a Poynting vector,

$$ S = \frac{1}{\mu_0} E \times B $$
which represents rate of transfer of electromagnetic energy per unit area: the interacting electric and magnetic fields carry momentum. We are therefore not justified in assuming the total momentum of the two charges should remain constant, nor that they both experience equal and opposite rates of change of momentum at any time since this ignores the continually changing momentum of the electromagnetic field. Although there is no magnetic field at charge \( q_2 \) due to the motion of charge \( q_1 \), there is a large volume in which the electric field \( E_2 \) of \( q_2 \) and the magnetic field \( B_1 \) of \( q_1 \) give rise to a Poynting vector with a momentum component in the opposite direction to the magnetic force on \( q_1 \). The exchange of momentum due to the magnetic interaction is not between \( q_1 \) and \( q_2 \) but between \( q_1 \) and the electromagnetic field. There is therefore no reason why Newton's third law should apply to the magnetic forces exerted on \( q_1 \) by \( q_2 \) and \( q_2 \) on \( q_1 \).

There are other considerations which must be taken into account also. The electrostatic force on \( q_2 \) due to \( q_1 \) is
\[
E_1 = \frac{q_1 q_2}{4\pi \varepsilon_0 r^2} \left( 1 - \frac{v^2}{c^2} \right),
\]
whilst that on \( q_1 \) due to \( q_2 \) is
\[
E_2 = \frac{q_1 q_2}{4\pi \varepsilon_0 r^2} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}},
\]
so these are not equal and opposite either. Then there is the question of how the two charges came to be in the positions of \( i_1 \) and \( i_2 \). They must have been interacting over extended paths in the past and the since any field changes travel at the speed of light the field at \( q_2 \) due to \( q_1 \) when their separation is \( r \) is that due to \( q_1 \) some time earlier when the separation was \( r' \neq r \). On top of this we must take account of the fact that as a result of these interactions the changes are experiencing accelerations and are therefore radiating energy and momentum away from the system completely. However it is the neglect of the momentum associated with the interacting electrostatic and magnetic
fields which is the dominant contribution to the paradox.

Whilst we are on the subject of currents and their magnetic fields, I would like to show you an example of an electric current which has no associated magnetic field and see how Messrs Biot and Savart get out of that! Suppose we charge up a parallel plate capacitor then disconnect it from the charging circuit and hold it in the open with neither of its terminals connected to anything. Now suppose the dielectric between the plates is leaky, i.e. let it have a small but finite conductivity. The capacitor will slowly discharge as a result of the current flowing through the dielectric.

\[ \text{Diagram of capacitor with arrows indicating current} \]

\[ P \]

Question: What is the magnetic field at P whilst the current flows?

Answer: Zero

This, at first sight, is a surprising answer. The Biot-Savart law says that currents give rise to magnetic fields and specifies the magnitude and direction, so how can the answer be zero? Well whilst the Biot-Savart law certainly says that currents give rise to magnetic fields it does not say that only currents give rise to magnetic fields. There are other sources of magnetic fields also and one of them happens to be present in our example.
The field at point P distant a from a long straight conductor carrying a current \( i \) is found by summing the contributions from all current elements \( i \, d\ell \) in the conductor

\[
B = \int dB = \frac{\mu_0}{4\pi} \int_{\theta=0}^{\theta=\pi} \frac{i \, d\ell \sin\theta}{r^2} = \frac{\mu_0 i}{4\pi} \int_{0}^{\pi} \frac{\csc^2\theta \, d\theta \, \sin\theta}{a^2 \csc^2\theta} = \frac{\mu_0 i}{4\pi a} \int_{0}^{\pi} \sin\theta \, d\theta = \frac{\mu_0 i}{2\pi a}
\]

If we add up all the products \( B \, ds \) around the line of \( B \) at distance \( a \) we find

\[
\oint B \, ds = \frac{\mu_0 i}{2\pi a} \cdot 2\pi a = \mu_0 i
\]

This result is actually very general and may be written

\[
\oint B \cdot ds = \mu_0 i \quad \text{(Ampère's circuital law)}
\]

where the "line integral" of \( B \) may be taken around any closed path threaded by the current \( i \). We can be precise with regard to what we mean by "threaded by the current \( i \)" by taking any surface \( S \) bounded by the path \( L \) of our line integral of \( B \). Then the current \( i \) threading the path \( L \) is the
current which passes through S.
e.g.

But now, since our paradox involves a capacitor let us put one in the circuit
and choose S to pass between its plates.

Clearly no current now passes through S but the charge on the plates is
increasing at a rate \( \frac{dq}{dt} = i \) and since \( \frac{q}{\varepsilon_0} \) lines of E originate on charge q
the total number of lines of E (or flux of E, \( \Phi_E \)) between the plates is
increasing at a rate \( \frac{d\Phi_E}{dt} = \frac{d}{dt}\left(\frac{q}{\varepsilon_0}\right) = \frac{i}{\varepsilon_0} \), and all of these lines of E pass
through the surface S.

Now we have \( i = \varepsilon_0 \frac{d\Phi_E}{dt} \) and \( \oint B \cdot ds = \varepsilon_0 \frac{d\Phi_E}{dt} \).
Hence so far as magnetic fields are concerned Ampère's conduction current \( i \) and Maxwell's displacement current \( \varepsilon_0 \frac{d\Phi_E}{dt} \) are equivalent. Conduction currents and displacement currents are equally effective in producing magnetic fields.

Now to return to our leaky charged capacitor.

The current \( i \) flows at the expense of the charge on the plates

\[
i = - \frac{dq}{dt}
\]

Lines of \( E \) are therefore disappearing at a rate \( \left| \frac{1}{\varepsilon_0} \frac{dq}{dt} \right| \)

\[
\frac{d\Phi_E}{dt} = \frac{1}{\varepsilon_0} \frac{dq}{dt} \quad \text{and} \quad \frac{d\Phi_E}{dt} \quad \frac{dq}{dt} \text{are both negative}
\]

\[
\frac{d\Phi_E}{\varepsilon_0 dt} = \frac{dq}{dt} = -i
\]

Thus the displacement current is equal and opposite to the conduction current and their magnetic fields cancel each other exactly. Our paradox is resolved.

There are many of these paradoxes in electromagnetism and the solution to most of them lies in the inclusion of some previously omitted contribution of the electromagnetic field. Time does not permit the discussion of any other tonight.