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LECTURE 3

ELECTROMAGNETIC FIELDS AND CIRCUITS: HOW ARE THEY RELATED?

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INTRODUCTION

The behaviour of electric and magnetic fields is governed by Maxwell's equation - a set of partial differential equations in space and time - together with a number of relations such as Ohm's law, which describe the properties of material media. In situations of constant geometry, in which the space-dependent aspects of the fields do not vary with time, it is possible to integrate over the spatial variables thereby arriving at lumped circuit elements, viz. resistance, capacitance and inductance. In general, this procedure (called lumping) is only approximate and to some extent arbitrary. This will be illustrated with reference to a very simple circuit, namely a single length of wire bent in a loop of arbitrary shape.

THE SIMPLEST CIRCUIT

Consider a piece of wire along which a non-equilibrium charge distribution is established, causing positive charge to accumulate at one end, and negative charge at the other end. Although the overall net charge is zero, an electric field $\mathbf{E}$ will have been established by the charge separation. Because the wire is a conductor, i.e. contains
mobile charge carriers, the electric field will cause a movement of charge, i.e. an electric current. The details of this are as follows: each charge carrier experiences a succession of collisions (due to its thermal motion) interspersed by periods of accelerated motion under the influence of $\mathbf{E}$. The net effect is an average motion, in the direction of $\mathbf{E}$, with a velocity proportional to $\mathbf{E}$.

This is called the drift velocity, $v_d$, and we write $v_d = \mu E$, where $\mu$ is the mobility of the charge carriers, assumed to be a constant for any particular conducting material. The field quantity associated with this transport of charge is the current density, $J$, defined to be the amount of charge crossing per unit time a unit area at right angles to $\mathbf{E}$. It may be seen that, for $n$ charge carriers per unit volume, each carrying charge $e$, there is the following relation between $J$ and $v_d$:

$$J = n e v_d \quad \text{and hence} \quad J = n e \mu E.$$ 

Provided that $n$ and $\mu$ stay constant (a reasonable assumption for many, but not all, situations) we may define new constants $\sigma$ and $\rho$ such that $J = \sigma \mathbf{E}$ or $\mathbf{E} = \rho J$. This relation is Ohm's Law, in microscopic form: $\sigma$ and $\rho$ are the conductivity and resistivity, respectively, properties of the material of the conductor.

We may also define the line integral of the electric field between the ends of the conductor, giving the potential difference:

$$V_{ab} = V_a - V_b = -\int_a^b \mathbf{E} \cdot d\mathbf{l} = \int_a^b \mathbf{E} \cdot d\mathbf{l}.$$ 

In terms of the current density $J$, this may be written:

$$V_{ab} = \int_a^b \rho J \cdot d\mathbf{l} = \int_a^b \rho J \ dl \quad \text{if} \quad d\mathbf{l} \text{ is taken in the direction of } J.$$ 

If $J$ is uniform over the cross-section of the wire $A$, we have that $AJ = I$, the total current carried, so that:
where \( R \), the \textit{resistance}, is a \textit{lumped} property of the circuit, arrived at by integrating over the space-dependent variables. (In general, for a non-uniform current density we have to perform a surface integral over the cross-section of the wire to get \( I = \int J \cdot dA \), and a line integral to obtain \( V_{ab} \). In any case, however complicated the actual calculation, we arrive at a definite value of \( R \), which depends only on the geometry and on the properties of the material of the conductor.) Thus, after this process of \textit{lumping} we arrive at Ohm's law in macroscopic form, viz. \( V = IR \), and a graphic representation as follows:

\[
\begin{array}{c}
V_a \\
\hline
R
\\
I
\\
V_b
\end{array}
\]

However, this is not an equilibrium situation: the charge distribution will even out and the potential energy will be dissipated unless an energy source is provided in order to maintain the potential difference \( V_{ab} \). (Furthermore, a closed circuit path must exist for a steady current \( I \) to flow.) This energy source (more precisely: energy converter) is of non-electrostatic origin so that unlike the electrostatic situation in which, around any closed path, \( \oint E \cdot dl = 0 \), we now have \( \oint E \cdot dl = \mathcal{E} \) where \( \mathcal{E} \) is the EMF around the closed path, i.e. the energy received per unit charge flowing around the circuit. Energy conservation then requires that:

\[
\mathcal{E} = IR
\]

Energy received = Energy dissipated \hspace{1cm} (per unit charge)

(Kirchhoff equation).
The source of EMF need not be localised, as in the case of a chemical battery: it may be distributed all around the circuit e.g. as a thermo-electric EMF, due to a temperature gradient, or, more familiarly, as an induced EMF due to a changing magnetic flux through the circuit. In general, however, it may be lumped also, i.e. represented by a localised object across which there is a rise in potential difference. We thus arrive at a simple, lumped representation of our simple circuit as shown below, which is the prototype of all DC circuits.

More complicated circuits differ simply in their topology and the possible inclusion of multiple sources of EMF and resistances lumped into separate circuit branches.

ELECTROSTATIC EFFECTS

Now that the source of EMF maintains the potential difference between the ends of the wire, and hence also the electric field, there is still an accumulation of charge of precisely the sort that we started with. The amount of charge \( q \) is related to the electric field by Gauss' law: its calculation entails the evaluation of a surface integral. The electric field also leads, via a line integral, to the potential difference \( V \). Both these integrals depend purely on the geometry of the circuit (and on the properties of any material media included in the circuit), and, for a fixed geometry, lead to a fixed constant of proportionality which relates \( q \) to the potential difference \( V \). This constant of proportionality is the capacitance \( C \), so that \( q = C V \), and is, once again, the result of lumping the electrostatic effects in the circuit.
The lumped representation that we have now is shown below, together with the modified Kirchhoff equations that follow.

\[ I - I' = \frac{dq}{dt} = C \frac{dV}{dt}; \]

\[ I' = \frac{V}{R}; \]

\[ I = \frac{V}{R} + C \frac{dV}{dt}. \]

We note that the circuit equations are no longer simple algebraic equations, but that a time-dependence has crept in, giving a differential equation. In the steady state, all the time-derivatives will be zero and we are back to the simple, lumped resistive circuit that we had before. But as soon as the EMF varies with time (e.g. when the circuit is turned on or off or when AC is applied), there will be important dynamical effects due to the capacitance, which cannot, in general, be ignored. A better understanding of these dynamical effects may be obtained by considering the energetics of the situation. In the steady state, energy is supplied by the source of EMF and is dissipated in the resistance of the circuit. There is, however, also electrostatic energy associated with the electric field \( W = \frac{1}{2} \varepsilon_0 E^2 \) per unit volume. The integral of this, over all the volume involved, is given by \( \frac{1}{2} CV^2 \) and represents the energy required to charge the capacitance of the circuit i.e. the energy required to establish the electric field everywhere in the space involved. When the source of EMF is first connected, this energy must be supplied by it; when it is turned off this energy will continue to supply the dissipation in the resistor until it runs out, i.e. until the charge accumulation disappears. (These processes are characterized by exponential variations with characteristic time constants given by the product RC.)
The above lumped representation of our simple circuit is really an over-simplification. We have lumped the resistance and the capacitance separately, while in reality they may both be present, in a distributed form all along the wire, as shown in the sequence of sketches below:

Each of these circuits is a better approximation to the situation, but it must be recognised that the process of lumping is in reality an approximation. In the limit, a correct description is only obtained with an infinite number of infinitesimal bits of $R$ and $C$. Naturally the dynamical effect discussed above will be more complicated. In the special case of the simplest geometry, namely constant $R$ and constant $C$ per unit length, the problem was posed and solved by Lord Kelvin, in 1855, in the belief that it applied to long transmission lines, such as the transatlantic cable which was then a very important practical issue. Although this belief turned out to be erroneous, as will be seen later, Lord Kelvin's analysis of the problem (of what is now known as the Kelvin Cable) is of more than historical interest and is sketched out below. (It has application in certain special situations, e.g. the base-emitter region of transistors, biological membranes, etc.)
\[ V + dV_1 \quad r \, dx \quad \quad V \quad r \, dx \quad V - dV_2 \]

\[ I \, dI \quad dI \quad I \, r \, dx \]

\[ dV_1 = (I + dI) \, r \, dx \]
\[ dI = c \, dx \frac{\partial V}{\partial t} \]
\[ dV_2 = I \, r \, dx \]

\[ \frac{dV_1 - dV_2}{dx^2} = \frac{\partial^2 V}{\partial x^2} = rc \frac{\partial V}{\partial t} \]

\[ \star \]

\[ r = \text{resistance per unit length} \]
\[ c = \text{capacitance per unit length} \]

This is the diffusion equation. Its solution shows that a transient signal propagates with its rise-time increasing as the square of the length of the cable, producing intolerable distortion by "slurring" successive signals. Had this been the correct physics of the situation the transatlantic cable would never have worked!

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ELECTROMAGNETIC EFFECTS

Another important effect which has to be considered is due to the magnetic fields which are present whenever there are currents flowing. It takes energy to establish a magnetic field \( W = \frac{1}{2}B^2/\mu_0 \) per unit volume and this energy is stored in the magnetic field. Again, while this has no consequences in the steady state, there will be dynamical effects whenever the current in the circuit changes. Any changing magnetic flux threading the circuit will induce an EMF, proportional to the rate of change. The magnetic field \( \mathbf{B} \) is related to the current via Ampere's law (a line integral) and the flux may be calculated from \( \mathbf{B} \) (via a surface integral). The result of these calculations is, once again, the lumping of the space-dependant variables into an overall constant of proportionality which relates the flux to the current. Thus \( \Phi = LI \) where \( L \) is the (self-)inductance of the circuit.
so that the induced EMF $\varepsilon'$, from Faraday's law, may be written as

$$\varepsilon' = -\frac{d\phi}{dt} = -L\frac{di}{dt}. \quad (The \ total \ magnetic \ energy \ becomes \ W = \frac{1}{2}LI^2.)$$

The self-induced EMF may be lumped in the circuit in series with the original, driving EMF. Alternatively, and more commonly, it is treated as a voltage drop and is taken over (with changed sign) to the other side of the Kirchhoff equation, as shown below.

$$\varepsilon + \varepsilon' = \varepsilon - L\frac{di}{dt} = RI \quad \varepsilon = RI + L\frac{di}{dt}$$

For the sake of simplicity, we have so far lumped $R$, $C$ and $L$ separately and this, of course, is not correct. In general all three effects are present at once, i.e. energy dissipation in resistance as well as energy storage in electric and magnetic fields ($C$ and $L$ respectively).

The simplest lumped representation of our circuit is, therefore, the one shown below, i.e. an R-L-C resonant circuit. This, however, is only an approximation, since we have lumped $R$, $L$ and $C$ separately.

It is clearly necessary, for a better approximation to split the circuit into different regions, each with its own $R$, $L$ and $C$. In the limit, this process recognises the distributed nature of the parameters and requires an infinite number of infinitesimal lumped components. To some extent these successive approximations are arbitrary, but one can (by calculation or by measurement) arrive at an infinite set of resonant frequencies (coupled characteristic modes) of higher and higher frequency, each with a corresponding lumped representation.
The validity of this process is easily demonstrated in practice: the response of our simple circuit to sinusoidal excitation may be explored with the aid of suitable measuring instruments. Alternatively, what we have chosen to do in a demonstration experiment is to connect the "circuit" (i.e. the length of wire of arbitrary shape) to a high-frequency transistor in a feedback configuration as shown below.

The transistor acts as a controlled source of EMF, the controlling signal being itself derived from the other end of the wire. The net result is a self sustaining oscillation (the energy losses being made up by the DC supply), at a frequency governed by the shape and size of the piece of wire, i.e. by its lumped inductance and capacitance. Several resonant modes may be excited at the same time, leading to a complicated waveform, showing that there is more to it than a single R - L - C circuit and that, in fact, higher modes can and do exist.

TRANSMISSION LINES

The simplest geometrical configuration is one which has a constant amount of L, R and C per unit length, as for example in the case of a pair of parallel wires (coaxial cable is another simple example). We now recognise that Lord Kelvin's mistake in analysing
the long cable problem was that he left out the effect of inductance. This was realised by Oliver Heaviside, that remarkable 19th Century scientist who, among many other important accomplishments, obtained the correct equations and thereby provided the scientific basis of all long-distance communications by cable. Heaviside's analysis of the problem is sketched out below, in a simplified form which neglects the resistance of the cable, i.e. neglects energy loss.

\[ V + dV = \ell \frac{d}{dt} (I + dI) \]

\[ dI = cdx \frac{\partial V}{\partial t} \]

\[ \ell = \text{inductance per unit length} \]

\[ c = \text{capacitance per unit length} \]

\[ \frac{dV_1 - dV_2}{dx^2} = \frac{\partial^2 V}{\partial x^2} = \ell c \frac{\partial^2 V}{\partial t^2} \]

This is the wave equation, in its simplest form. Its solutions show that signals propagate along the cable with a uniform velocity, given by \( \frac{1}{\sqrt{\ell c}} \). For parallel wires the constants are calculable, giving:

\[ \ell = (\mu_0/\pi) \text{ arcosh } (D/d) \]

\[ c = (\pi \varepsilon_0)/\text{arcosh } (D/d) \]

so that \( 1/\sqrt{\ell c} = 1/\sqrt{\mu_0 \varepsilon_0} = 3 \times 10^8 \text{ m s}^{-1} \), the velocity of light!

The wave-like propagation of signals can be demonstrated with a long piece of (coaxial) cable, showing propagation delays and reflections from open and short-circuited ends.

The effects of finite resistance are easily included, along with finite leakage resistance of the insulating material of the cable. The resultant wave equation, known as the telegraphist's equation, is slightly more complicated and will not be dealt with here. Its solutions show attenuation and dispersion of the waves and lead to the practical limitations recognised by the early telegraphists. Heaviside's treatment of the problem remains the basis upon which long-distance communications by cable have developed.
RADIATION RESISTANCE

There is yet another effect which has not been dealt with so far. It is most easily brought into evidence by opening up the ends of our cable into what will be recognised as an antenna. Changing currents flowing in the vertical sections give rise to magnetic fields, which in turn produce a return flux of electric field, i.e. a displacement current. The changing displacement current will, in turn produce magnetic fields, and so forth. The net result is that self-sustaining electromagnetic fields are produced and radiated into space, in the form of electromagnetic waves.

![Diagram of electromagnetic waves](image)

The energy carried away by these waves is completely lost from the circuit, as if it were dissipated. Hence, in order to account for all the energy terms (stored, dissipated or radiated), we must include an extra lumped component which, for the above reason is resistive in nature, and is called the radiation resistance. It is usually quite small except in the special circumstances required to produce a good antenna, i.e. a good radiating device.

CONCLUSION

DC circuits, with steady currents flowing, may be exactly represented by lumped resistances and sources of EMF. When dealing with changing EMFs and currents, the energy stored in electric and
magnetic fields manifests itself. These effects become increasingly important when rapid rates of change are involved, i.e. at high frequencies. They may be taken into account by lumped circuit elements, viz. capacitance and inductance, but the process of lumping is then only approximate. In the limit, when the spatially distributed nature of inductance and capacitance are taken into account, the circuits regain the partial differential equation aspect of the field equations from which they arose in the first place. The historically famous (and practically important) problem of transmission lines illustrates this conclusion.